Final exam Robust Control

29 October 2024, 11:45–13:45

The exam consists of 4 exercises. Please write clearly and provide clear motivations for your answers. You get 4 points for free and the maximum possible number of points is 40. Your grade is equal to the number of points divided by 4.

1 $(1 + 2 + 3 + 3 = 9$ points) **Stabilization**

Consider the system Σ given by

$$\dot{x}(t) = Ax(t) + Bu(t),\ y(t) = Cx(t).$$

Here, $u(t) \in \mathbb{R}^m$ is the control input, $x(t) \in \mathbb{R}^n$ is the state and $y(t) \in \mathbb{R}^p$ is the measured output. In addition, consider controllers Γ of the form

$$\dot{w}(t) = Kw(t) + Ly(t),\ u(t) = Mw(t) + Ny(t),$$

where $w(t) \in \mathbb{R}^{n_c}$ is the state of the controller.

(a) Write down a state-space representation of the feedback interconnection $\Sigma \times \Gamma$.

(b) Formulate the problem of stabilization by dynamic output feedback.

(c) Find a state observer for Σ. Show that, under suitable conditions, the estimation error tends to 0 as $t \to \infty$.

(d) Formulate necessary and sufficient conditions for the existence of a stabilizing controller Γ for Σ, and explain how to compute one.

2 $(2 + 3 + 4 = 9$ points) H_2 **optimal control**

Consider the system Σ of the form

$$\dot{x}(t) = Ax(t) + Bu(t) + Ed(t),\ z(t) = Cx(t) + Du(t).$$

Here, $u(t)$ is the control input, $d(t)$ the disturbance input, and $z(t)$ the performance output.

(a) Give the precise definition of the H_2 optimal control problem by state feedback.

(b) Assume that (A, B) is stabilizable and the problem is in standard form. Explain how to compute the optimal performance. Moreover, state sufficient conditions under which there exists an optimal state feedback, and explain how to compute one.

(c) Now consider the special case that

$$\dot{x}(t) = ax(t) + bu(t) + d(t),\ z(t) = u(t),$$

where all variables are scalars. Assume that $b \neq 0$. For all values of a, compute the optimal performance. Also compute an optimal state feedback, or provide a convincing argument that it does not exist.
Hint: Distinguish between the cases that $a < 0$, $a = 0$ and $a > 0$.

3 (2 + 3 + 4 = 9 points) H_∞ norm, bounded real lemma

Consider the system

$$\dot{x}(t) = Ax(t) + Bu(t),\ y(t) = Cx(t) + Du(t),$$

with transfer matrix $G(s) = C(sI - A)^{-1}B + D$.

(a) Provide the definition of the H_∞ norm of the transfer matrix $G(s)$.

(b) Let $\gamma > 0$. During the lectures, we have seen the bounded real lemma: A is Hurwitz and $||G||_\infty < \gamma$ if and only if there exists a symmetric matrix Y such that

$$Y > 0, \quad \begin{bmatrix} A^\top Y + YA + C^\top C & YB + C^\top D \\ B^\top Y + D^\top C & D^\top D - \gamma^2 I \end{bmatrix} < 0. \tag{1}$$

Assume that there exists a symmetric Y satisfying (1). Prove that A is Hurwitz.

(c) Assume that there exists a symmetric Y satisfying (1). Prove that $||G||_\infty < \gamma$.
Hint: Multiply the second matrix inequality in (1) from left by $\begin{bmatrix} x(t)^\top & u(t)^\top \end{bmatrix}$ and from right by its transposed.

4 (4 + 5 = 9 points) Small gain theorem

Consider the system Σ_i given by

$$\dot{x}_i(t) = A_i x_i(t) + B_i u_i(t),\ y_i(t) = C_i x_i(t) + D_i u_i(t),$$

for $i = 1, 2, 3$. Assume that A_i is Hurwitz and the transfer matrix

$$G_i(s) = C_i(sI - A_i)^{-1}B_i + D_i$$

satisfies $||G_i||_\infty < \gamma_i$, where $\gamma_i > 0$ for $i = 1, 2, 3$. In what follows, we study the interconnected system $\Sigma_1 \times \Sigma_2 \times \Sigma_3$ obtained by setting $u_1 = y_3$, $u_2 = y_1$ and $u_3 = y_2$.

(a) We call the interconnected system $\Sigma_1 \times \Sigma_2 \times \Sigma_3$ *well-posed* if $I - D_1D_2D_3$ is invertible. Show that $\Sigma_1 \times \Sigma_2 \times \Sigma_3$ is well-posed if $\gamma_1\gamma_2\gamma_3 < 1$.

(b) Show that $\Sigma_1 \times \Sigma_2 \times \Sigma_3$ is asymptotically stable if $\gamma_1\gamma_2\gamma_3 < 1$.

End of exam, cheers!